Decoding challenge

Assessing the practical hardness of syndrome decoding for code-based cryptography

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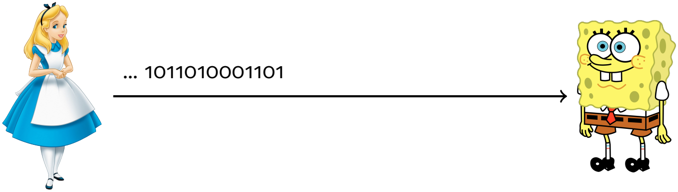
All you ever wanted to know about code-based crypto

Public Key Cryptography



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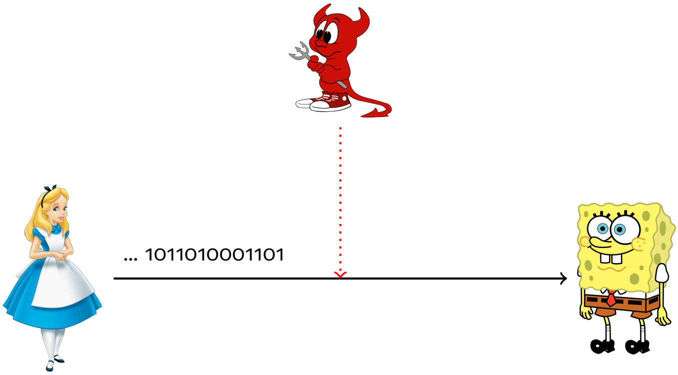

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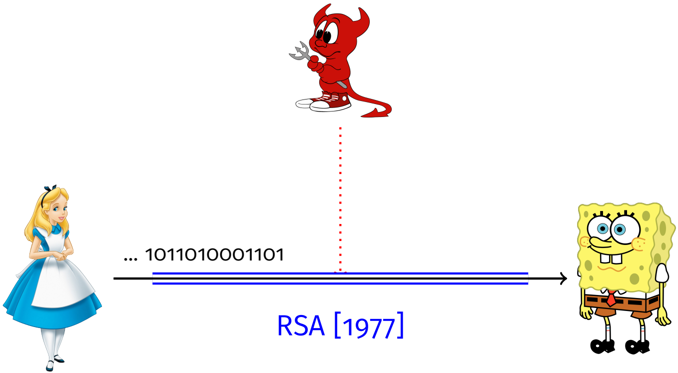

[1994]

+

+

$$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

... 1011010001101

~~RSA [1977]~~

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Post-Quantum Cryptography

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Lattice | <u>Codes</u> | Hash | Multivariate | Isogenies

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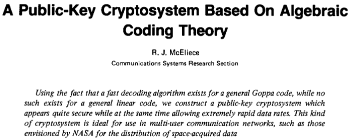
1978, Robert McEliece: [McE78]


A Public-Key Cryptosystem Based On Algebraic Coding Theory

R. J. McEliece
Communications Systems Research Section

Using the fact that a fast decoding algorithm exists for a general Goppa code, while no such exists for a general linear code, we construct a public-key cryptosystem which appears quite secure while at the same time allowing extremely rapid data rates. This kind of cryptosystem is ideal for use in multi-user communication networks, such as those envisioned by NASA for the distribution of space-acquired data

Definition (Code)

An $[n,k]_{\mathbb{F}_q}$ linear **code** $\mathcal{C}$ is a linear subspace of $\mathbb{F}_q^n$ of dimension k.

Definition (Decoder)

A **decoder** for the code $\mathcal{C}$ is a function

$$\Phi_{\mathcal{C}} : \mathbb{F}_q^n \to \mathcal{C} \cup \{?\}.$$

We say that $\Phi_{\mathcal{C}}$ can decode up to t errors if

$$\forall c \in \mathcal{C}, \forall e \in \mathbb{F}_q^n, \qquad |e| \leq t \quad \Rightarrow \quad \Phi_{\mathcal{C}}(c+e) = c.$$

Definition (Generator matrix)

A **generator matrix** of a code $\mathcal{C}$ is a matrix $\mathbf{G} \in \mathbb{F}_q^{k \times n}$ such that:

$$\mathcal{C} = \{x\mathbf{G} \mid x \in \mathbb{F}_q^k\}.$$

Definition (Parity-check matrix)

A **parity-check matrix** of a code $\mathcal{C}$ is a matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$ such that:

$$\mathcal{C} = \{y \in \mathbb{F}_q^n \mid \mathbf{H}y^\intercal = 0\}.$$

Example (Repetition Code)

$$\begin{array}{ccc} \mathbb{F}_2 & \to & \mathbb{F}_2^3 \\ 0 & \mapsto & (0,0,0) \\ 1 & \mapsto & (1,1,1) \end{array}$$

Example (Decoder)

```
if |x| <= 1:
    return 0
else:
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if |x| <= 1:
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$$G = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Main idea: how hard is it to decode up to t errors?

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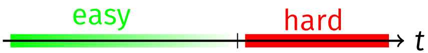

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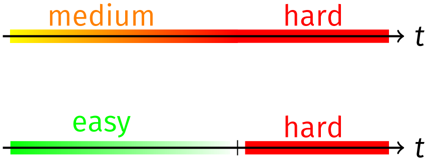

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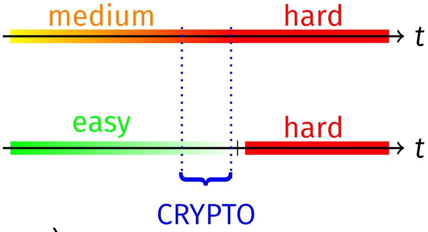

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How to design a code-based cryptosystem?

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Receipe:

KeyGen()	Enc(m)	Dec(c)
$\mathbf{G}_{\mathsf{sk}} \stackrel{\$}{\leftarrow} \mathcal{F}$ $\mathbf{G}_{\mathsf{pk}} \leftarrow \mathrm{Shake}(\mathbf{G}_{\mathsf{sk}})$	$e \stackrel{\$}{\leftarrow} \mathbb{F}_q^n$, s.t. $\lvert e\rvert = t$ $c \leftarrow m\mathbf{G}_{\mathsf{pk}} + e$	$m \leftarrow \Phi_{\mathcal{F}}(\mathbf{G}_{\mathsf{sk}}, c)$

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- the shaker shakes well enough!

How could Eve break this scheme? 2 possibilities:

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Remark: Hypothesis 1 depends on the choice of the family of codes $\mathcal{F}$ and the shaker, while Hypothesis 2 is generic!

Some examples

- Examples of choices of $\mathcal{F}$:
  - Goppa codes [Original McEliece];
  - Reed Solomon codes [Nie86] (broken by [SS92]);
  - QC-MDPC codes [BIKE];
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- Examples of shakers:
  - row scrambler;
  - columns isometry (permutation);
  - subfield subcode;
  - adding random columns...

Syndrome Decoding

Let $\mathcal{C}$ be an $[n, k]$ linear code of parity-check matrix $\mathbf{H}$.
Let $y \in \mathbb{F}_q^n$ and $s = y\mathbf{H}^\intercal \in \mathbb{F}_q^k$ (the **syndrome** of y).
The following problems are equivalent.

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3. Find an error e such that $e\mathbf{H}^\intercal = s$ and $|e| \leq t$.

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The Syndrome Decoding Problem - $\mathrm{SD}(q, n, R, W)$

Instance: $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$,
$s \in \mathbb{F}_q^{n-k}$.

Output: $e \in \mathbb{F}_q^n$ such that $|e| = w$ and $e\mathbf{H}^\mathsf{T} = s$,

where $k \triangleq \lceil Rn \rceil$, $w \triangleq \lceil Wn \rceil$.

Theorem (NP-completeness)

The Syndrome Decoding problem is NP-complete. [BMvT78]

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Conjecture (average case)

Decoding n^{ε} errors is hard on average $\forall \varepsilon > 0$. [Ale11]

Binary Syndrome Decoding Problem

From now on, we focus on the binary case $q = 2$.

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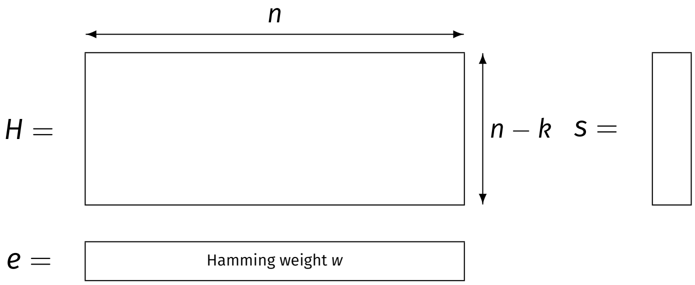

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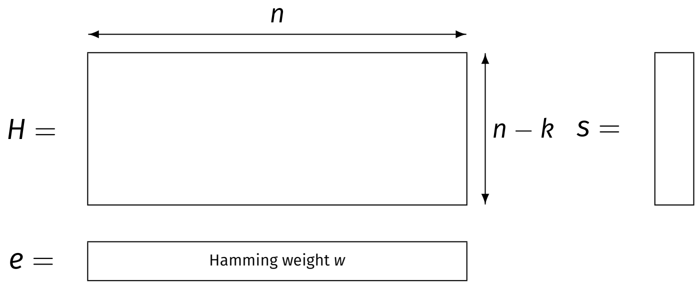

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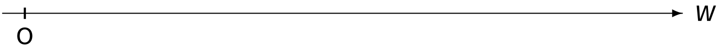
The next slides of this section are reproduced from Nicolas Sendrier's MOOC "Code Based Cryptography" with his authorization.

Fix n and k, let w grow:



Number of solutions

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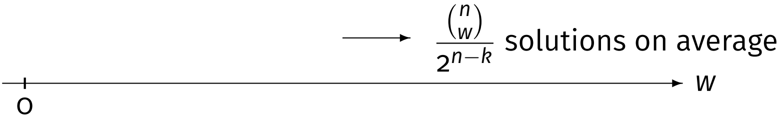

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at most one solution $\longleftarrow$ $\longrightarrow$ $\frac{\binom{n}{w}}{2^{n-k}}$ solutions on average

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0 — d_{GV} — w

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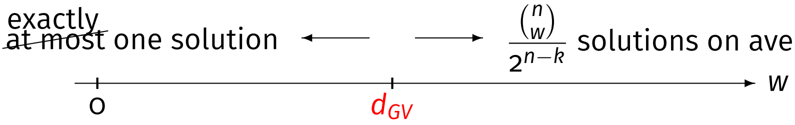

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We expect $\approx \max\left(1, \binom{n}{w}/2^{n-k}\right)$ solutions.

Problem:
find w columns of H adding to s (modulo 2)

$$H = \overbrace{\boxed{\begin{matrix} h_1 & h_2 & \cdots & h_n \end{matrix}}}^{n} \; n-k \qquad s = \boxed{}$$

Exhaustive search

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Enumerate all w-tuples $(j_1, j_2, \cdots, j_w)$ such that

$$1 \leq j_1 < j_2 < \ldots < j_w \leq n.$$

Check whether

$$s + h_{j_1} + h_{j_2} \cdots + h_{j_w} = 0.$$

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Cost: *about* $\binom{n}{w}$ column operations.

Remark: we obtain all solutions.

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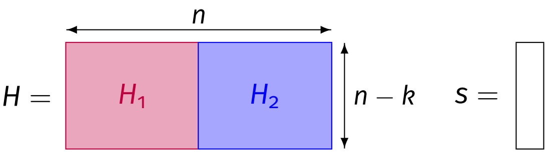

BIRTHDAY ALGORITHM

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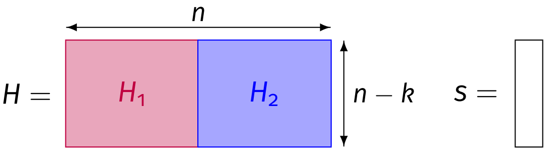

Idea: Split H into two equal parts and enumerate the two following sets

$$\mathcal{L}_1 = \left\{ e_1 H_1^T, |e_1| = \frac{w}{2} \right\} \text{ and } \mathcal{L}_2 = \left\{ s + e_2 H_2^T, |e_2| = \frac{w}{2} \right\}$$

If $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$, we have solution(s): $s + e_1 H_1^T + e_2 H_2^T = 0$

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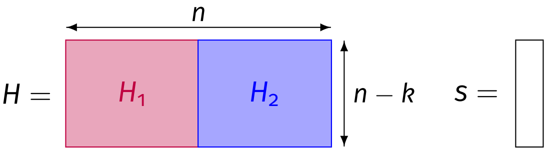

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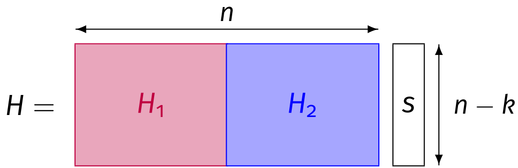
If $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$, we have solution(s): $s + e_1 H_1^T + e_2 H_2^T = 0$

Cost: Requires *about* $2L + L^2/2^{n-k}$ column operations, where $L = |\mathcal{L}_1| = |\mathcal{L}_2|$

Birthday algorithm

$$\boxed{\text{Compute } \mathcal{L}_1 \cap \mathcal{L}_2 = \left\{ e_1 H_1^T \mid |(|e_1) = \tfrac{w}{2} \right\} \cap \left\{ s + e_2 H_2^T \mid |(|e_2) = \tfrac{w}{2} \right\}}$$



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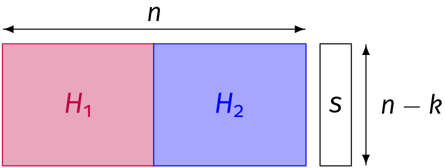
$\quad x \leftarrow e_1 H_1^T \; ; T[x] \leftarrow T[x] \cup \{e_1\}$

$H =$ 

Total cost: $\binom{n/2}{w/2}$

$|\mathcal{L}_1|$

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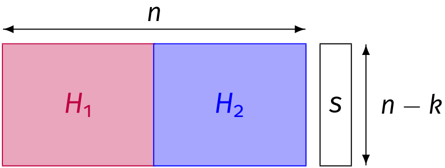
$H =$ 

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for all e_1 of weight w/2
    x ← e_1 H_1^T ; T[x] ← T[x] ∪ {e_1}
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    for all e_1 ∈ T[x]
        I ← I ∪ {(e_1, e_2)}
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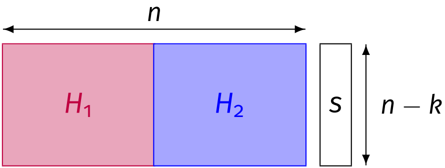
$H =$ 

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return I
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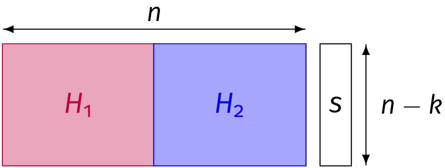
$H =$



Total cost: $\binom{n/2}{w/2} + \binom{n/2}{w/2} + \dfrac{\binom{n/2}{w/2}^2}{2^{n-k}}$

$|\mathcal{L}_1| \qquad |\mathcal{L}_2| \qquad \dfrac{|\mathcal{L}_1| \cdot |\mathcal{L}_2|}{2^{n-k}}$

Birthday algorithm

One particular error of Hamming weight w splits evenly with probability

$$\mathcal{P} = \frac{\binom{n/2}{w/2}^2}{\binom{n}{w}}$$

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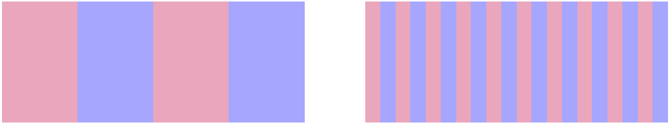
We may have to repeat with H divided in several different ways



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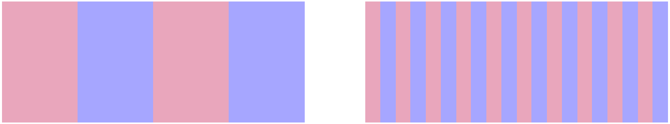

or more generally by picking the two halves randomly

Repeat $1/\mathcal{P}$ times to get most solutions. **Cost:** $O\left(\sqrt{\binom{n}{w}}\right)$.

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For any invertible $U \in \{0,1\}^{(n-k)\times(n-k)}$ and any permutation matrix $P \in \{0,1\}^{n\times n}$

$$\left(eH^T = s\right) \Leftrightarrow \left(e'H'^T = s'\right) \text{ where } \left\{ \begin{array}{ccc} H' & \leftarrow & UHP \\ s' & \leftarrow & sU^T \\ e' & \leftarrow & eP. \end{array} \right.$$

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Proof:
$$\begin{aligned} e'H'^T &= (eP)(UHP)^T \\ &= (eP)P^T H^T U^T \\ &= eH^T U^T \\ &= sU^T \\ &= s'. \end{aligned}$$

Idea: Perform a Gaussian Elimination and hope that all the errors are in positions corresponding to the identity part!

$$H' = UHP = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & \end{array}\right] \quad \text{and } s' = sU^T = \left[\begin{array}{c} \\ \\ \\ \end{array}\right]$$

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$$H' = UHP = \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{n-k} & \phantom{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}} \end{array}\right] \quad \text{and } s' = sU^T = \left[\phantom{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}}\right]$$

possible if the first $n-k$ columns of HP are independent

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REPEAT:

1- Pick a permutation matrix P

Prange's algorithm

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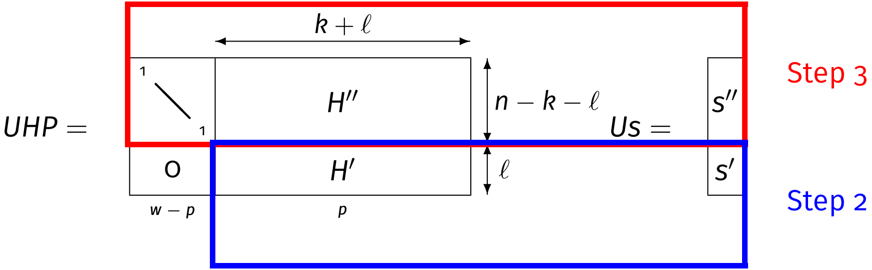
Cost of one iteration: $\mathcal{K} = n(n-k)$ column operations.
Success probability: $\mathcal{P} = \binom{n-k}{w} / \binom{n}{w}$.

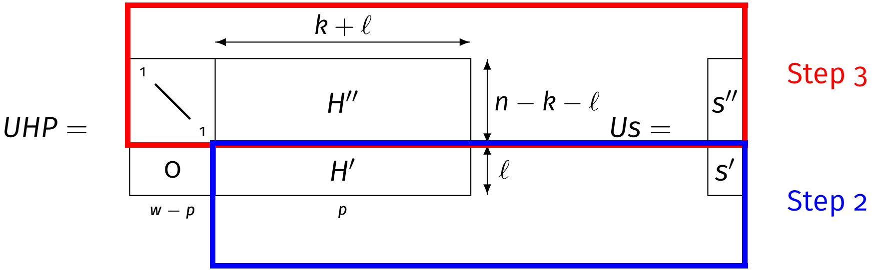
Total cost = $\mathcal{K}/\mathcal{P}$.

Stern and Dumer's algorithm

$$
UHP = \begin{array}{cc}
 & \overleftrightarrow{\quad k+\ell \quad} \\
\left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & H'' \\ \hline 0 & H' \end{array}\right] & \begin{array}{l} n-k-\ell \\ \\ \ell \end{array} \\
\begin{array}{cc} w-p & \qquad\qquad p \end{array} &
\end{array}
\qquad
Us = \begin{bmatrix} s'' \\ \hline s' \end{bmatrix}
$$


$UHP =$
$k + \ell$
1
1
H''
$n - k - \ell$
$Us =$
s''
Step 3
0
H'
ℓ
s'
$w - p$
p
Step 2

Stern and Dumer's algorithm



Repeat:
1. Permutation + partial Gaussian elimination
2. Find many e' such that $|e'| = p$ and $H'e' = s'$
3. For all good e', test $|s'' + H''e'| \leq w - p$

Step 2 is Birthday Decoding (or whatever is best);
Step 3 is (a kind of) Prange;
Total cost is minimized over ℓ and p.

Iteration cost: $\mathcal{K} = n(n-k-\ell) + 2\sqrt{\binom{k+\ell}{p}} + \frac{\binom{k+\ell}{p}}{2^\ell} + \frac{\binom{k+\ell}{p}}{2^\ell}$

Iteration cost: $\mathcal{K} = \underbrace{n(n-k-\ell)}_{\text{Gaussian elimination}} + 2\sqrt{\binom{k+\ell}{p}} + \frac{\binom{k+\ell}{p}}{2^\ell} + \frac{\binom{k+\ell}{p}}{2^\ell}$

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Iteration cost: $\mathcal{K} = n(n-k-\ell) + 2\sqrt{\binom{k+\ell}{p}} + \frac{\binom{k+\ell}{p}}{2^\ell} + \frac{\binom{k+\ell}{p}}{2^\ell}$

Gaussian elimination

Birthday decoding

Final check

Success probability: $\mathcal{P} = \frac{\binom{k+\ell}{p}\binom{n-k-\ell}{w-p}}{\binom{n}{w}}$.

Total cost = $\mathcal{K}/\mathcal{P}$, minimized over p and ℓ.

More advanced algorithms

- Improved Birthday Decoding: overlapping support.
- Representations.
- Recursive Birthday Decoding.
- Decoding One Out of Many.
- Nearest Neighbour approach.

- **Theoretical** asymptotic exponent

  Best algorithm solves $\mathrm{SD}(n, W, R)$ in $2^{c \cdot n}$ operations with

1962	c = 0.121	[Pra62]
1988	c = 0.117	[Ste88, Dum89]
2011	c = 0.112	[MMT11]
2012	c = 0.102	[BJMM12]
2017	c = 0.095	[MO15, BM17]
2018	c = 0.089	[BM18]

  for $w = \mathrm{d}_{\mathrm{GV}}$ and worst choice of k.

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- **Practical** complexity?

The Decoding Challenge

decodingchallenge.org

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decodingchallenge.org

Home Generic problems Documentation Contact NIST-like problems

Welcome to the code-based challenges webpage!

The purpose of this webpage is to assess the **practical hardness** of problems in coding theory.

The Challenges

The following challenges are currently available.

Generic problems. First we consider the two main problems on which Hamming-weight code-based cryptography mainly relies.

> Challenge: Syndrome Decoding

Given a matrix $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$, a vector $\mathbf{s} \in \mathbb{F}_2^{n-k}$ and an integer $w \leq n$, find a vector $\mathbf{e} \in \mathbb{F}_2^n$ of Hamming weight $|\mathbf{e}| \leq w$ such that $\mathbf{H}\mathbf{e}^\top = \mathbf{s}^\top$. Here we will focus especially on the case with rate $R = 0.5$ and w close to the Gilbert-Varshamov bound.

> Challenge: Finding Low Weight Codewords

In a random linear code of size $n = 1280$ and rate $R = 0.5$, there exists on average a unique codeword of weight 144 (the Gilbert-Varshamov bound) and finding it should requires at least 2^{128} operations with the best known algorithms. Finding words of higher weight is easier. The goal of this challenge is to find codewords with a weight as close as possible to the GV-bound.

NIST-like problems. We propose challenges with the same parameter settings as the main cryptographic schemes proposed for the NIST standardization process for post-quantum cryptography. For now, we propose two such challenges in Hamming metric. In both cases, the goal is to assess the hardness of generic decoding, not to find distinguishers on the codes. Therefore we propose random linear codes with the same rate and error weight as the corresponding NIST candidates.

Latest news

20-08-2019. Announcement of the challenge at Crypto 2019 conference

12-08-2019. All challenges are now online

21-07-2019. Website is online.

12-07-2019. Contact email is active.

Launched in August 2019 by Aragon, Lavauzelle and L.

- **Goal:**
  - assess the practical complexity of problems in coding theory;
  - motivate the implementation of ISD algorithms;
  - increase the confidence in code-based crypto.

The Decoding Challenge

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  - assess the practical complexity of problems in coding theory;
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- **Concept:**
  - 4 categories of challenges;
  - instances of increasing size;
  - a hall of fame.

- 2 generic problems
  - Syndrome Decoding $k/n = 0.5$ and $w = \mathrm{d}_{\mathrm{GV}}$
  - Finding the Lowest Codeword
    for $k/n = 0.5$ and n of cryptographic size
- 2 problems based on schemes in the NIST competition
  - Goppa-McEliece $k/n = 0.8$ and $w = (n-k)/\log_2(n)$
  - QC-MDPC $k/n = 0.5$ and $w = \sqrt{n}$

Questions raised by implementation

Based on previous work from Landais, Sendrier, Meurer and Hochbach,

and recent work from Vasseur, Couvreur, Kunz and L.

- Choice of parameters p, ℓ, ε ... must be integers!
- Random shuffle vs. Canteaut-Chabaud.
- Birthday algorithm: sort vs. hash table.
- Allowing overlap?
- Early abort?
- ...

It's not just about asymptotic exponents anymore!

decodingchallenge.org

- **How to contribute?**
  - Solve some challenges!
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Current leader of the Hall of Fame:
Valentin Vasseur, $n = 450$ (for SD)
$\simeq 2^{47}$ operations (Dumer).

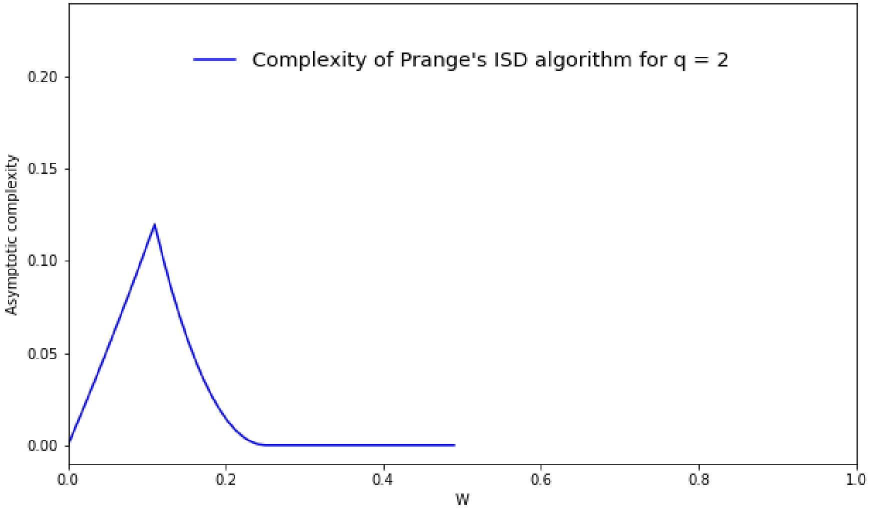
You dream to read your name in a Hall of Fame?
This is the chance of a lifetime!

Future challenges

- We intend to propose other categories of challenges
  - rank-metric Syndrome decoding;
  - q-ary Syndrome Decoding in Hamming metric;
  - q-ary Syndrome Decoding in Hamming metric with large weight.

q-ary Syndrome Decoding

for $R = 1/2$:


Complexity of Prange's ISD algorithm for q = 2
Asymptotic complexity
0.00
0.05
0.10
0.15
0.20
0.0
0.2
0.4
0.6
0.8
1.0
W

Binary vs. ternary Decoding Challenge

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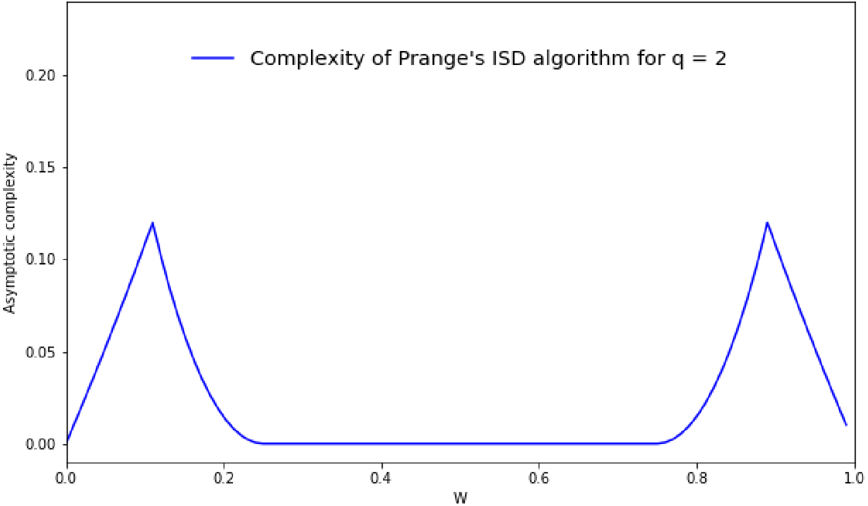

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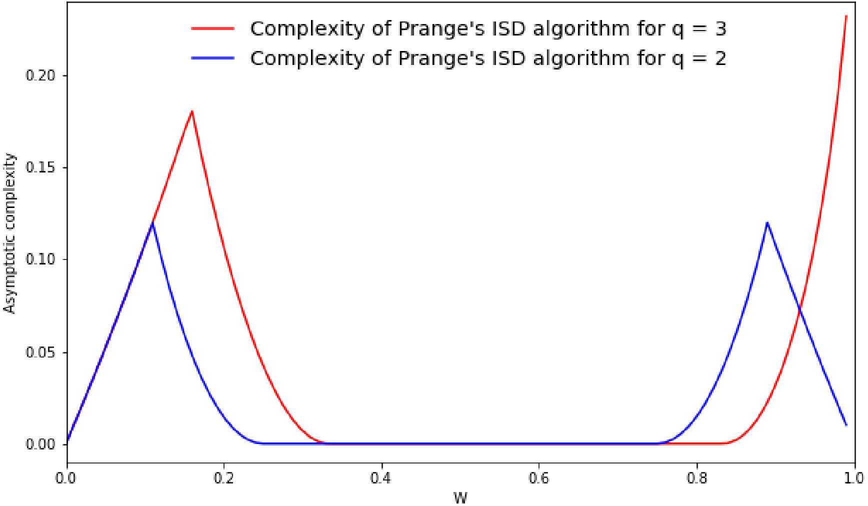
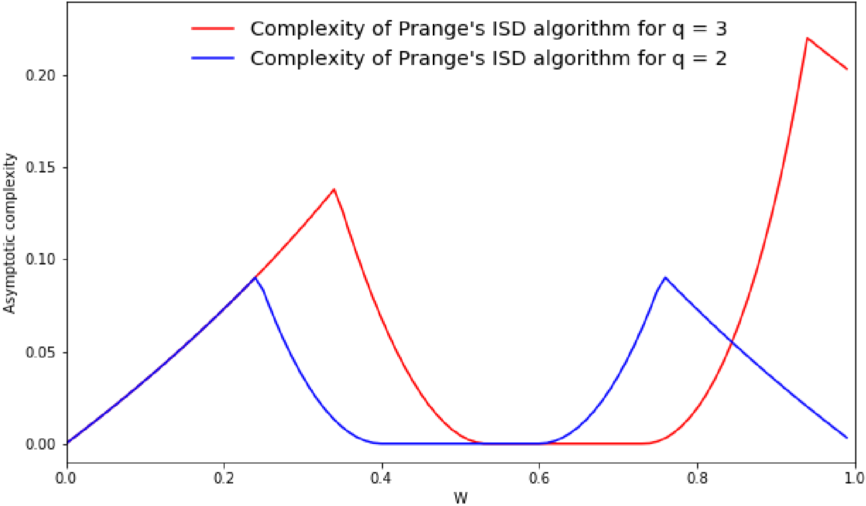


for $R = 1/5$:


Complexity of Prange's ISD algorithm for q = 3
Complexity of Prange's ISD algorithm for q = 2
Asymptotic complexity
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0.05
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$$\binom{n}{d}(q-1)^d = q^{n-k}.$$

- Worst case complexity for Prange's algorithm is reached for

$$R = 1 - \log_q(q-1) \text{ and } W = 1.$$

for $q = 3$ this is $R = 0.369$.

"Ternary Syndrome Decoding with Large Weight",
Bricout, Chailloux, Debris-Alazard and L., SAC 2019

- Motivation: Wave signature scheme [DST19].

Doing better than Prange?

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Given $k + \ell$ vectors $\mathbf{h}_i \in \mathbb{F}_3^\ell$ and a target vector $s \in \mathbb{F}_3^\ell$,
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This can be solved using Wagner's algorithm [Wag02].

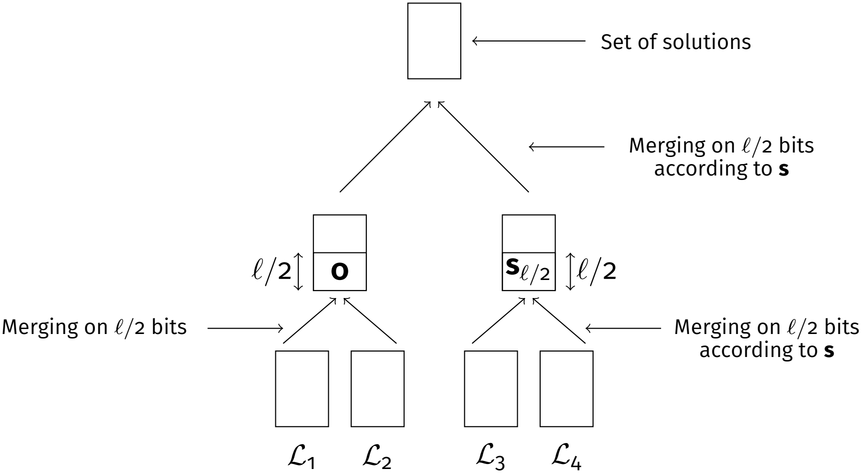

Figure: Wagner's algorithm with $a = 2$.

- Using Wagner's algorithm with a floors and $L = 3^{\ell/a}$ solutions can be solved in amortize time $O\left(3^{\ell/a}\right)$.

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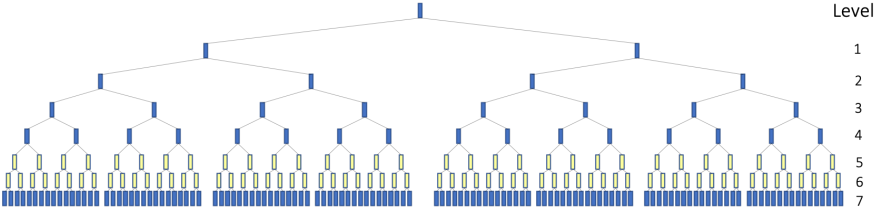
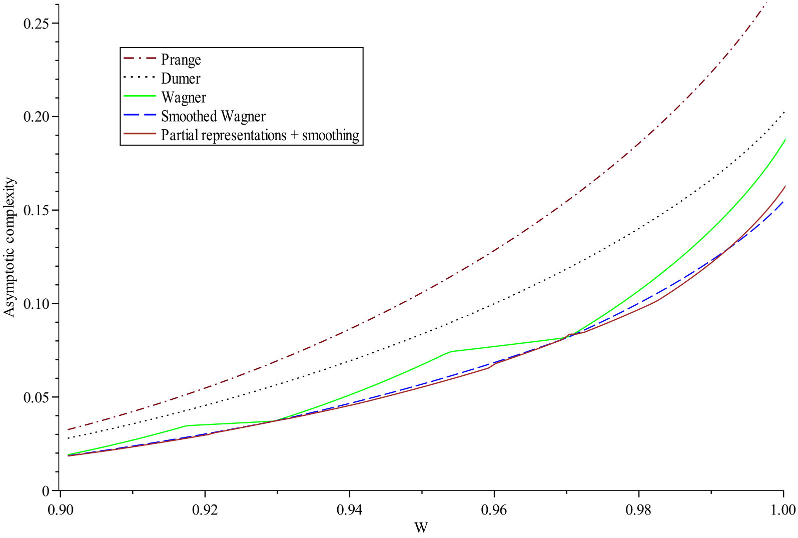
Remark: When $q \to \infty$, all ISD algorithm become equivalent to Prange's algorithm [Can17].

Our algorithm [BCDL19]

- 7 floors
- Blue = “left-right” splits (no representations)
- Yellow = representations
- Badly-formed elements at floor 4 and 5



Results (R = 0.5) [BCDL19]



HARDEST INSTANCES FOR $q = 3$ [BCDL19]

Algorithm	$q = 2$	$q = 3$ and $W > 0.5$
Prange	0.121 (R = 0.454)	0.369 (R = 0.369)
Dumer/Wagner	0.116 (R = 0.447)	0.269 (R = 0.369)
BJMM/our algorithm	0.102 (R = 0.427)	0.247 (R = 0.369)

Table: Best exponents with associated rates.

Hardest instances for $q = 3$ [BCDL19]

Algorithm	$q = 2$	$q = 3$ and $W > 0.5$
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Table: Best exponents with associated rates.

Algorithm	$q = 2$	$q = 3$ and $W > 0.5$
Prange	275	44
Dumer/Wagner	295	83
BJMM/Our algorithm	374	99

Table: Minimum input sizes (in kbits) for a time complexity of 2^{128}.

Concluding remarks

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Thank you for your attention!

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